Visual Artifacts

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The discussion began with Dr. Bill Gropp introducing the concept of visual artifacts in numerical solutions. He presented examples of errors that appeared to be significant to the human eye, but that were well below the error criteria for the problem, and did not impact the quality of the numerical solution.

The discussion then focused on defining a model problem where visual artifacts could be examined explicitly.

Model Problem

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1 Introduction

At the ICASE Workshop on Heterogeneous Boundary Conditions a general optics problem that allows interference was suggested for study. The large-scale interference pattern that develops is quite sensitive to small perturbations in the boundary conditions. Hence it seemed ideal for testing and observing errors due to grid interface effects introduced by domain decomposition methods. Although the problem specification is somewhat arbitrary, it is necessary to be specific in order to compare results because it is expected that several researchers will explore this problem.

2 Error Measurements

The interference that we wish to test lends itself to error analysis using both a visual as well as more standard numerical acceptance criteria. The more standard numerical criteria are

- propagation error

- L_1 error

- L_2 error

- L_∞ error

By propagation error we mean the phase difference between the computed location of the wave front and the exact location of the same wave front.

Visually this problem gives rise to an interference pattern that can be compared for sharpness as well as location. We are interested in seeing these differences across the artificial interfaces introduced by the decomposition.

3 Motivation

Dr. Bill Gropp suggested we examine a problem that had visual meaning in its errors, to allow studying the types of errors introduced between refined and coarse meshes at a more intuitive level. He suggested an optics interference problem.

4 Optical Interference

Fermats' principle of least time is recast in Feynman et. al. [1] briefly from a quantum-dynamical perspective. By considering "rays of light" as photons, the ray path can be considered a sum of the individual paths of the photons. The ray path is then defined by the probability of each photon taking different paths.

Upon encountering a barrier that contains a wide slit, a wave will continue through the slit almost undisturbed and geometric optics is a good model for ray behavior. But when the slit is sufficiently reduced in size, the choices for photon paths are truncated, and the probability distribution is altered, affecting the geometry of the wave front as it passes through the slit.

This behavior is more easily understood by considering the simpler, less accurate, but more intuitive, Huygens principle which states that "all points on a wavefront can be considered as point sources for the production of spherical secondary wavelets. After a time the new position of the wavefront will be the surface of tangency to these secondary wavelets", as described in Halliday and Resnick [2]. Considering this simplified wave theory of light, a barrier in a wave path with a slit on the order of the wave length will cause diffraction. That is, the end points allowed to pass through the slit will no longer have symmetrical wavelets on either side, and they will bend at the ends.

Placing two slits aside each other will replicate the limiting behavior twice, and the resulting wave patterns will interact, causing interference. The interference pattern will be visible, and dependent upon the original wave frequency. This interference pattern is quite sensitive to phase errors, so that the choice of grid sizes influences the solution behavior. This is the interesting aspect of this problem.

5 The Interference Pattern

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u.$$

Here Δ is the Laplacian. In the positive quadrant place two slits along the x-axis at locations x_1 and x_2, with

$$d = x_2 - x_1.$$

Here x_1 and x_2 are the mid-points of the two slits. Then for any point in the quadrant, $P = (x_p, y_p)$, let d_1 be the distance from x_1 to P, and d_2 be the distance from x_2 to P.

Waves will arrive at P out of phase due to the difference in the path lengths d_1 and d_2. The maximum interference will occur when

$$|d_2 - d_1| = m\lambda$$

where λ is the original incident plane wavelength and m is a nonnegative integer. The minimum, of course, occurs at the half distances $(m + \frac{1}{2})$.

The size of our slits can now be specified. They should be at most of width λ. A slit width less than or equal to the incident wavelength is sufficiently small to diffract the wave on a visible scale. This problem is interesting because generation of the large scale pattern depends on how accurately the small scale dynamics has been captured about the slits.

6 Geometry

Here we assume that the problem has been normalized so that

$$x_l = 0,$$

$$x_r = 1,$$

$$y_b = 0,$$

$$y_t = 1,$$

We arbitrarily choose $x_1 = \frac{1}{2} - 5\lambda$, $x_2 = \frac{1}{2} + 5\lambda$. This geometry is illustrated on the following page in Fig. 1.

7 Initial Conditions

The initial region, including all boundaries except the slits, should be quiescent ($u = 0$).

Prescribe a plane wave described by the following function:

$$sin\frac{2\pi}{\lambda}(y - vt) \tag{7.1}$$

where λ is the wavelength and v is the phase velocity. This impacts our domain along the slits $[\frac{1}{2} - 5.5\lambda, \frac{1}{2} - 4.5\lambda]$ and $[\frac{1}{2} + 4.5\lambda, \frac{1}{2} + 5.5\lambda]$ of the x-axis for all time $t > 0$. Here, the equation reduces to

$$sin\frac{2\pi}{\lambda}(-vt). \tag{7.2}$$

The phase velocity is known, allowing radiative boundary conditions to be defined. This is done in the section 9.
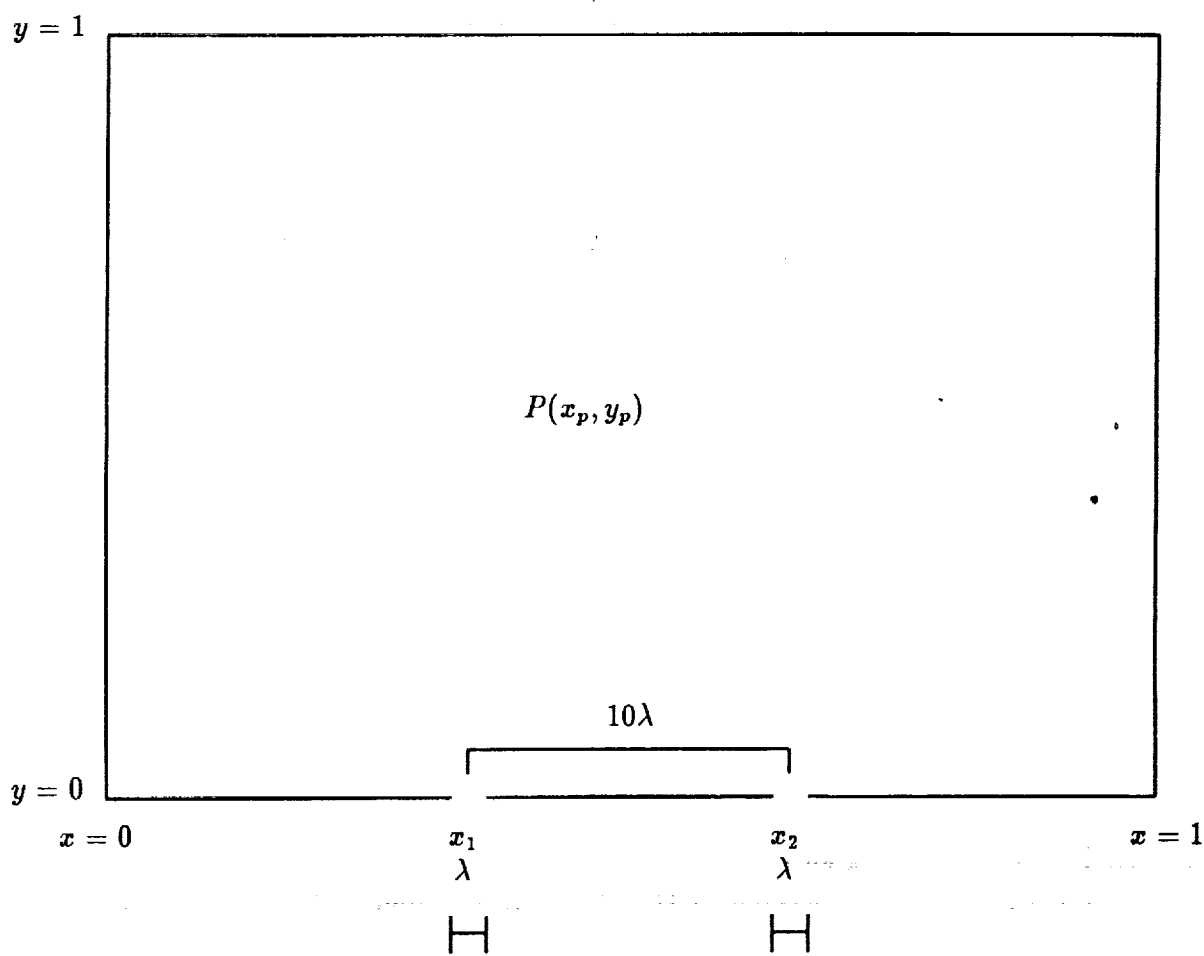
Figure 6.1: Problem Domain

8 Exact Solution

The exact interference pattern is the superposition of the two waves. For any point $P(x, y)$ in the domain, the travel time to P will be different from the two slits. Let the time from slit x_i be t_i. Then the value at P will be the sum of the P_i where

$$P_i(x, y) = sin\frac{2\pi}{\lambda}(-v(t - t_i))$$

for $(t - t_i) \geq 0$ and zero otherwise. We note that the distance from slit x_i to the point $P(x, y)$ is

$$d_i = ((x - x_i)^2 + y^2)^{\frac{1}{2}}.$$

9 Boundary Conditions

All boundaries, except the slits which are prescribed with the incoming plane wave, evolve with the solution. Any workable boundary conditions can be applied on these boundaries, with the goal that these boundary conditions should influence the interference pattern as little as possible.

For our prescribed incident plane wave we have the exact solution. However, prescription of these exact boundary values on our numerical approximation of the solution could cause the numerical approximation to degrade. Hence we recommend radiative boundary conditions because we know the phase velocity exactly and hope that the numerical phase velocity is quite close to the correct one. Then an open boundary condition can be used that advects the interference pattern out of the domain by advancing the wave equation,

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial n} = 0$$

where n is the direction normal to the boundary.

10 Domain Decomposition Method

Our purpose in examining this test problem is to measure directly the effect of mesh refinement and the resulting mesh interfaces on a known wave that is sensitive to phase errors, while concurrently being able to visually display a meaningful picture of the effects of the refinement-induced error on the solution.

The Coarse mesh must be able to adequately represent the interference pattern for the visual comparisons. It should be no larger than $\Delta x = \Delta y = \frac{\lambda}{4}$, and may need to be smaller. The workshop suggested $\frac{\lambda}{10}$. We suggest that $\frac{\lambda}{4}$, $\frac{\lambda}{10}$ and $\frac{\lambda}{20}$ all be tested. However, these values are only *apriori* suggestions as we have not yet worked this problem.

The refined mesh must be able to adequately capture the diffraction behavior, so that the plane wave front "bends" as it passes through the slit. Given that the coarse and refined

meshes are sufficiently accurate, the phase errors introduced during the problem solution will be a function of the sound speed on the two grids plus the coarse/refined grid interaction errors.

Due to the geometry of the problem, non-adaptive local uniform mesh refinement is adequate for the domain decomposition.

11 Discretization

We suggest a second order in space and time Leap Frog/Hopscotch discretization method should be used. We anticipate beginning with $\lambda = 0.1$, and $\lambda = 0.05$.

12 Analysis

We expect that calculations for boundary errors and the ratio of mesh refinements will be analyzed. These can be done both analytically and by comparison to an everywhere fine mesh.

References

[1] R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures On Physics Mainly Mechanics, Radiation, and Heat*, Addison-Wesley, (26-8), 1963.

[2] D. Halliday, and R. Resnick, *Fundamentals of Physics*, John Wiley and Sons, 1970.